It is interesting to note from result (6·17) that the first component of $\mathbf{x}_j^{(1)}$ is $(z_j - c_j)$ or (Δ_j), which is always used to decide the optimality.

Note. The greatest advantage of treating the objective function as one of the constraints is that, $z_j - c_j$ or (Δ_j) for any $\mathbf{a}_j$ not in the basis can be easily computed by taking the product of first row of $\mathbf{B}_1^{-1}$, with $\mathbf{a}_j^{(1)}$ not in the basis, that is,

$$\Delta_j = z_j - c_j = (\text{first row of } \mathbf{B}_1^{-1}) \times \mathbf{a}_j^{(1)} \text{ not in the basis.}$$

(VII) The $(m+1)$-component solution vector $\mathbf{X}_\mathbf{B}^{(1)}$ is given by

$$\mathbf{X}_\mathbf{B}^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{b}^{(1)} \qquad \text{...(6·18)}$$

or

$$\mathbf{X}_\mathbf{B}^{(1)} = \begin{bmatrix} 1 & \mathbf{C}_\mathbf{B}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 \times 0 + \mathbf{C}_\mathbf{B}(\mathbf{B}^{-1}\mathbf{b}) \\ 0 \times 0 + \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C}_\mathbf{B}\mathbf{X}_\mathbf{B} \\ \mathbf{X}_\mathbf{B} \end{bmatrix} = \begin{bmatrix} z \\ \mathbf{X}_\mathbf{B} \end{bmatrix} \qquad [\text{because } \mathbf{X}_\mathbf{B} = \mathbf{B}^{-1}\mathbf{b},\ \mathbf{C}_\mathbf{B}\mathbf{X}_\mathbf{B} = z]$$

Thus,

$$\mathbf{X}_\mathbf{B}^{(1)} = \begin{bmatrix} \mathbf{C}_\mathbf{B}\mathbf{X}_\mathbf{B} \\ \mathbf{X}_\mathbf{B} \end{bmatrix} = \begin{bmatrix} z \\ \mathbf{X}_\mathbf{B} \end{bmatrix} \quad \textbf{(Note)} \qquad \text{...(6·19)}$$

In (6·19), it has been observed that $\mathbf{X}_\mathbf{B}^{(1)}$ is a basic solution (not necessarily feasible, because z may be negative also) for the matrix equation (6·7)′ corresponding to the basis matrix $\mathbf{B}_1$. Also, the first component of $\mathbf{X}_\mathbf{B}^{(1)}$ immediately gives the value of the objective function while the second component $\mathbf{X}_\mathbf{B}$ gives exactly the basic feasible solution to original constraint system $\mathbf{AX} = \mathbf{b}$ corresponding to its basis matrix $\mathbf{B}$. Thus the result (6·19) is of great importance.

Now the results of this section are applied for computational procedure of revised simplex method.

6.5. TO OBTAIN INVERSE OF INITIAL BASIS MATRIX AND INITIAL BASIC FEASIBLE SOLUTION

6.5·1. When No Artificial Variables are Needed

As discussed in **section 6·4**, the inverse of initial basis matrix $\mathbf{B}_1$ is given by

$$\mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C}_\mathbf{B}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \qquad \text{...(6·20)}$$

But, the initial basis matrix $\mathbf{B}$ for the original problem is always $(m \times m)$ identity matrix $(\mathbf{I}_m)$. It should be noted that $\mathbf{I}_m$ always appears in $(\mathbf{AX} = \mathbf{b})$ (if it is not so, it can be made to appear in $\mathbf{A}$ by introducing the artificial variables).

Since $\mathbf{B} = \mathbf{I}_m = \mathbf{B}^{-1}$, $\quad \mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C}_\mathbf{B}\mathbf{I}_m \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}$ $\quad$ or $\quad \mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C}_\mathbf{B} \\ 0 & \mathbf{I}_m \end{bmatrix}$

Furthermore, if after ensuring that all $b_i \geq 0$, only the slack variables are needed and the initial basis matrix $\mathbf{B} = \mathbf{I}_m$ appears, then

$$c_{B1} = c_{B2} = c_{B3} = \ldots = c_{Bm} = 0,\ i.e.\ \mathbf{C}_\mathbf{B} = \mathbf{0}.$$

Thus, (6·20) becomes

$$\mathbf{B}_1^{-1} = \begin{bmatrix} \mathbf{1} & : & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & : & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = \mathbf{I}_{m+1}$$

Thus, it can be concluded that the inverse of the initial basis matrix $\mathbf{B}$ will be $\mathbf{B}_1^{-1} = \mathbf{B}_1 = \mathbf{I}_{m+1}$ to start with the revised simplex procedure.

Then, the initial basic solution becomes

$$\mathbf{X}_\mathbf{B}^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{b}^{(1)} = \mathbf{I}_{m+1}\,\mathbf{b}^{(1)} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \qquad \text{...(6·21)}$$

which is feasible.

After obtaining the initial basis matrix inverse $\mathbf{B}_1^{-1} = \mathbf{I}_{m+1}$ and an initial basic feasible solution to start with the revised simplex procedure, we need to construct the starting revised simplex table.

6.5·2. To Construct the Starting Table in Standard Form –I

Since x_0 (= z) should always be in the basis, the first column $\beta_0^{(1)}$ (= $\mathbf{e}_1$) of initial basis matrix inverse $\mathbf{B}_1^{-1} = \mathbf{I}_{m+1}$ will not be removed at any subsequent iteration. The remaining column vectors of $\mathbf{B}_1^{-1}$ will be $\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_m^{(1)}$.

The last column in the revised simplex table will be $\mathbf{X}_k^{(1)} = \begin{bmatrix} z_k - c_k \\ \mathbf{X}_k \end{bmatrix} = \begin{bmatrix} \Delta_k \\ \mathbf{X}_k \end{bmatrix}$ where k is predetermined by the formula

$$\Delta_k = \min \Delta_j \text{ (for those } j \text{ for which } \mathbf{a}_j \text{ is not in } \mathbf{B}_1).$$

Note. If there is a tie, we can use smallest index j which is an arbitrary rule but computationally strong.

Finally, it is concluded that only the column vectors $\mathbf{e}_1, \beta_1^{(1)}, \beta_2^{(1)} \ldots, \beta_m^{(1)}$ of $\mathbf{B}_1^{-1}, \mathbf{X}_B^{(1)}$ and $\mathbf{X}_k^{(1)}$ will be needed to construct the revised simplex table.

Now the starting table for revised simplex method can be constructed as follows. Also, for convenience, form an additional table for those $\mathbf{a}_j^{(1)}$ which are not in the basis and will be useful to determine the required Δ_j's.

Starting Table in Standard Form–I

Table 6·1

Variables in the basis	$\mathbf{e}_1$	$\mathbf{B}_1^{-1}$: $\beta_1^{(1)}$	$\beta_2^{(1)}$	...	$\beta_m^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
z	1	0	0	...	0	0	$z_k - c_k$
x_{B1}	0	1	0	...	0	b_1	x_{1k}
x_{B2}	0	0	1		0	b_2	x_{2k}
:	:	:	:		:	:	:
:	:	:	:	:	:	:	:
x_{Bm}	0	0	0	...	1	b_m	x_{mk}

Table (6·1)′

Additional table for those $\mathbf{a}_j^{(1)}$ which are not included in the $\mathbf{B}_1^{-1}$ of starting table.

We now proceed to demonstrate how the computational procedure discussed so far can be applied to solve the practical problems.

Q. Describe the revised simplex procedure for solving a L.P.P. **[Meerut (L.P.) 90]**

6.6. APPLICATION OF COMPUTATIONAL PROCEDURE : STANDARD FORM–I

Now apply the computational procedure of revised simplex method to solve numerical problems of linear programming. All necessary steps involved in this procedure can be easily understood by solving a simple type of problem. All the necessary steps are explained in a systematic order by applying each of them to the following illustrative example so that each step could be followed more easily without any trouble.

Illustrative Example

Example 1. *Solve the following simple linear programming problem by revised simplex method.* *Max* $z = 2x_1 + x_2$, *subject to* $3x_1 + 4x_2 \le 6$, $6x_1 + x_2 \le 3$, *and* $x_1, x_2 \ge 0$.

[Kanpur 96; Delhi (B. Sc. Math.) 93]

Solution. Step 1. ***Express the given problem in Standard Form–I.***

After ensuring that all $b_i \ge 0$ and transforming the objective function of original problem for maximization of z (if necessary), introduce non-negative slack variables to convert the *restrictive inequalities to equations*. It should be remembered that the objective function is also treated as if it were the first constraint equation.

Thus, the given problem is transformed to the following suitable form,

$$\begin{aligned} z - 2x_1 - x_2 \qquad\qquad &= 0 \\ 3x_1 + 4x_2 + x_3 \qquad &= 6 \\ 6x_1 + x_2 \qquad + x_4 &= 3 \end{aligned} \qquad \text{...(i)}$$

Step 2. ***Construct the starting table in revised simplex form.***

Now proceed to obtain the initial basis matrix $\mathbf{B}_1$ as an identity matrix and complete all the columns of starting revised simplex table except the last column $\mathbf{X}_k^{(1)}$ (which can be filled up in *Step 5* only).

Applying this step, the system (i) of constraint equations can be expressed in the following matrix form.

$$\begin{array}{ccccc} \mathbf{e}_1\ (=\beta_0^{(1)}) & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)}(=\beta_1^{(1)}) & \mathbf{a}_4^{(1)}(=\beta_2^{(1)}) \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

Here the columns $\beta_0^{(1)}$, $\beta_1^{(1)}$ and $\beta_2^{(1)}$ will constitute the basis matrix $\mathbf{B}_1$ (whose inverse is also $\mathbf{B}_1$, because $\mathbf{B}_1 = \mathbf{I}_3$ here). Now starting revised simplex table can be constructed as follows :

Table 6·2

Variables in the basis	$\mathbf{B}_1^{-1}$			$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
	e_1 (z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$		
z	1	0	0	0	
$x_{B1} = x_3$	0	1	0	6	
$x_{B2} = x_4$	0	0	1	3	

Additional Table 6·2′

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_2^{(1)}$
−2	−1
3	4
6	1

First Iteration

Step 3. ***Computations of*** $\Delta_j = z_j - c_j$ ***for*** $\mathbf{a}_1^{(1)}$ ***and*** $\mathbf{a}_2^{(1)}$.

Applying the formula : $\Delta_j =$ (first row of $\mathbf{B}_1^{-1}$) $\times$ ($\mathbf{a}_j^{(1)}$ not in the basis),

$$\Delta_1 = (\text{first row of } \mathbf{B}_1^{-1}) \times \mathbf{a}_1^{(1)} = (1, 0, 0)(-2, 3, 6) = [1 \times (-2) + 0 \times 3 + 0 \times 6] = -2$$

$$\Delta_2 = (\text{first row of } \mathbf{B}_1^{-1}) \times \mathbf{a}_2^{(1)} = (1, 0, 0)(-1, 4, 1) = [1 \times (-1) + 0 \times 4 + 0 \times 1] = -1 .$$

Remark. Instead of computing each required Δ_j separately, we can also compute them simultaneously in single step as follows :

$$\{\Delta_1, \Delta_2\} = (\text{first row of } \mathbf{B}_1^{-1})\,[\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}] = [1, 0, 0] \begin{bmatrix} -2 & -1 \\ 3 & 4 \\ 6 & 1 \end{bmatrix}$$

or $$\{\Delta_1, \Delta_2\} = \begin{bmatrix} 1 \times (-2) + 0 \times 3 + 0 \times 6 \\ 1 \times (-1) + 0 \times 4 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \{-2, -1\}$$

which gives the values $\Delta_1 = -2$, $\Delta_2 = -1$ as obtained earlier.

Step 4. ***Apply test of optimality.***

Now apply usual simplex rule to test the starting solution ($x_1 = x_2 = 0$, $x_3 = 6$, $x_4 = 3$) for optimality.

Since Δ_1, Δ_2 obtained in *step 3* are both negative, so the starting basic feasible solution is not optimal. Hence we must proceed to determine the entering vector $\mathbf{a}_k^{(1)}$.

Step 5. ***Determination of the 'entering vector'*** $a_k^{(1)}$.

To determine the vector $\mathbf{a}_k^{(1)}$ entering the basis matrix at the subsequent iteration, find such value of k for which the criterion : $\Delta_k = \min. \{\Delta_j\}$ *for those j for which* $\mathbf{a}_j^{(1)}$ *are not in the basis is satisfied*

So, in this example, we have $\Delta_k = \min. [\Delta_1, \Delta_2] = \min [-2, -1] = -2 = \Delta_1$

$$\therefore \quad \Delta_k = \Delta_1 \Rightarrow k = 1.$$

Hence $\mathbf{a}_1^{(1)}$ enters the basis. This indicates that the corresponding variable x_1 will enter the solution.

Now, in order to find the leaving vector in *Step 7*, first compute $\mathbf{X}_k^{(1)}$ for $k = 1$ in the next step.

Step 6. ***Compute column vector*** $\mathbf{X}_k^{(1)}$ ***(for*** $k = 1$***).***

Since $\mathbf{X}_k^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_k^{(1)} = \mathbf{I}_{m+1}\,\mathbf{a}_k^{(1)}$ therefore, $\mathbf{X}_1^{(1)} \equiv \mathbf{a}_1^{(1)} = (-2, 3, 6)$.

Now complete the last column $\mathbf{X}_k^{(1)}$ of starting *Table 6·2* by writing $\mathbf{X}_1^{(1)} = \mathbf{a}_1^{(1)} = (-2, 3, 6)$ in that column. So the starting Table 6·2 grows to the following form.

Table 6·3

Variables in the basis	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_3^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_4^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
z	1	0	0	0	-2
x_3	0	1	0	6	3
x_4	0	0	1	3	6

Step 7. ***Determination of the leaving vector*** $\beta_r^{(1)}$***, given the entering vector*** $a_1^{(1)}$***.***

The vector $\beta_r^{(1)}$ to be removed from the basis is determined by using the **minimum ratio rule** (similar to that of ordinary simplex method) to find the value of suffix r for predetermined value of k $(= 1)$. *i.e.*,

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0 \text{ for } k = 1\right] = \min_i \left[\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right] = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{6}{3}, \frac{3}{6}\right] = \frac{3}{6}.$$

$$\therefore \quad \frac{x_{Br}}{x_{r1}} = \frac{x_{B2}}{x_{21}} \Rightarrow r = 2 \text{ (Equating the suffixes on both sides } ({}_{r1} = {}_{21}) \text{ find } r = 2.)$$

The value of r thus obtained shows that the vector $\beta_2^{(1)}$ must leave the basis.

Table 6·4

Variables in the basis	$\mathbf{e}_1$	$\beta_1^{(1)}$ $(a_3^{(1)})$	$\beta_2^{(1)}$ $(a_4^{(1)})$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$	Min. ratio rule : min. $\left(\frac{\mathbf{X}_B}{\mathbf{X}_1}\right)$
z	1	0	0	0	-2	
$x_{B1} = x_3$	0	1	0	6	3	6/3
$x_{B2} = x_4$	0	0	1	3	[6]	3/6 ←
			↓ Leaving vector $\beta_2^{(1)}$		↑ Key column	

Note. It is interesting to note that the entire process of *Step 7* can be more conveniently performed by adding one more column in *Table 6·3*, for 'minimum ratio rule' (as we have seen in ordinary simplex method). *In table 6.4*, we observe that the number 6 in the column $\mathbf{X}_1^{(1)}$ comes out to be the *'key element* or *pivot element*. So we must bring unity at its place and zero at all other places of this column $\mathbf{X}_1^{(1)}$ in order to determine the transformed table from which the new (improved) solution can be read off.

Remark. If the $\min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right]$ is attained for more than one value of i, the resulting basic feasible solution will be degenerate.

So, in order to ensure that cycling will never occur, we shall use our usual techniques to resolve the degeneracy.

Step 8. ***Determination of the improved solution by transforming Table 6·4.***

In order to bring uniformity with the ordinary simplex method, adopt the simple matrix transformation rules which are easier for hand computations. Here the intermediate coefficient matrix can be written as :

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
R_1	0	0	0	-2
R_2	1	0	6	3
R_3	0	1↓	3	[6]